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On the existence of a smallest compact support of a seminorm and a linear mapping

by Jorge Alberto Barroso[1] and Leopoldo Nachbin[2]

[1] *Instituto de Matemática, Universidade Federal do Rio de Janeiro, 21944 Rio de Janeiro, RJ Brasil*
[2] *Department of Mathematics, University of Rochester, Rochester NY 14627, U.S.A. and Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, 22290 Rio de Janeiro, RJ Brasil*

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ABSTRACT

Let $\mathscr{S}$ be a vector subspace of the vector space $\mathscr{C}(E;F)$ of all continuous mappings of a completely regular space E to a real or complex Hausdorff locally convex space F. A compact subset K of E is a support of a seminorm p on $\mathscr{S}$ if, whenever f lying in $\mathscr{S}$ vanishes on some neighborhood of K in E, then $p(f)=0$. In the special case that $p=|\phi|$ or $p=\|u\|$, where ϕ is a linear form on $\mathscr{S}$, or more generally, u is a linear mapping of $\mathscr{S}$ into a normed linear space, we say that K is a support of ϕ, or of u, respectively. Sufficient conditions are given in order that, if p has some compact support, then p has a smallest compact support (Proposition 4); and, $\mathscr{S}$ being endowed with a locally convex topology, that every continuous p has a smallest compact support (Corollary 7). Such results apply to the vector subspace $\mathscr{C}^{(m)}(U;F)$ of $\mathscr{C}(U;F)$ of all mappings of U to F that are continuously m-differentiable, say in the Hadamard, or Fréchet, or other noteworthy senses, where U is a nonvoid open subset of a real locally convex space E; or even to a more general situation, subsuming known examples with an additional nuclearity condition, such as in [10] and other references in the Bibliography (Example 8).

INTRODUCTION

Fix a nonvoid open subset U of $\mathbf{R}^n$ $(n=1,2,\ldots)$, and $m=0,1,\ldots,\infty$. Let $\mathscr{D}^{(m)\prime}(U)$ be the vector space of all distributions of order at most m on U, namely the dual space of the locally convex space $\mathscr{D}^{(m)}(U)$ of all continuously m-differentiable scalar valued functions on U with compact supports. Each element of $\mathscr{D}^{(m)\prime}(U)$ has the smallest closed support in U. On the other hand, letting $\mathscr{E}^{(m)}(U)$ be the locally convex space of all continuously m-differentiable scalar valued functions on U, its dual space $\mathscr{E}^{(m)\prime}(U)$ is naturally identified with the vector subspace of $\mathscr{D}^{(m)\prime}(U)$ of those elements whose smallest closed supports are compact. If $m=0$, we have similar, but more general, considerations for $\mathscr{K}(U)=\mathscr{D}^{(0)}(U)$, $\mathscr{K}'(U)=\mathscr{D}^{(0)\prime}(U)$ on a locally compact space U, and for $\mathscr{C}(U)=\mathscr{C}^{(0)}(U)$, $\mathscr{C}'(U)=\mathscr{C}^{(0)\prime}(U)$ on a completely regular space U. In passing from $\mathbf{R}^n$ to a real Hausdorff locally convex space E, and a nonvoid open subset U of E, the literal analogues of $\mathscr{D}^{(m)}(U)$, $\mathscr{D}^{(m)\prime}(U)$ vanish if E is infinite dimensional; whereas those of $\mathscr{C}^{(m)}(U)$, $\mathscr{C}^{(m)\prime}(U)$, or even variants of them occuring in infinite dimensions but coinciding with them in finite dimensions, remain meaningful. Since the method of continuously m-differentiable partitions of unit, available on $\mathbf{R}^n$, no longer is at our disposal on an infinite dimensional E, a slightly different approach is called for to prove existence of the smallest compact support. The purpose of this article is to provide sufficient conditions for existence of the smallest compact support (Proposition 4 and Corollary 7); and to apply them to the continuously m-differentiable case, which presents itself in some interesting variations (Example 8). We deal here with seminorms and linear mappings, but skip multilinear mappings, and polynomials.

NOTATION 1. Unless stated otherwise, we denote by E a completely regular space, by F a real or complex Hausdorff locally convex space, by $\mathscr{C}(E;F)$ the vector space of all continuous mappings of E to F endowed with the compact-open topology, and by $\mathscr{S}$ a fixed vector subspace of $\mathscr{C}(E;F)$. Usually p denotes a seminorm on $\mathscr{S}$. We write $\mathscr{C}(E)=\mathscr{C}(E;\mathbf{K})$ if $\mathbf{K}=\mathbf{R}$ or $\mathbf{K}=\mathbf{C}$.

DEFINITION 2. A compact subset K of E is *a support of* p, or *supports* p, if $p(f)=0$ whenever $f\in\mathscr{S}$ and f vanishes on a neighborhood of K in E; then p is *supported by* K. In case K is the smallest support of p, then K is *the support* of p. If ϕ if a linear form on $\mathscr{S}$, then $p=|\phi|$ is a seminorm on $\mathscr{S}$, to which the present definition applies; we then say that K is *a support of* ϕ, or *supports* ϕ, and that ϕ is *supported by* K. These considerations apply, more generally, to a linear mapping u of $\mathscr{S}$ to a normed space, and $p=\|u\|: f\in\mathscr{S}\mapsto\|u(f)\|\in\mathbf{R}$.

DEFINITION 3. A *multiplier* of $\mathscr{S}$ in $\mathscr{C}(E)$ is any $f\in\mathscr{C}(E)$ such that $f\mathscr{S}\subset\mathscr{S}$. Such multipliers form a subalgebra $\mathscr{M}(\mathscr{S})$ of $\mathscr{C}(E)$ containing the unit 1. If $\mathscr{S}$ is a subalgebra of $\mathscr{C}(E)$ containing the unit 1, then $\mathscr{M}(\mathscr{S})=\mathscr{S}$.

PROPOSITION 4. Assume that $\mathscr{M}(\mathscr{S})$ separates points of E in the following sense: *(sp)* if $x_0,x_1\in E$, $x_0\neq x_1$, there is $f\in\mathscr{M}(\mathscr{S})$ such that $f=i$ on a neighborhood of x_i in E, for $i=0,1$. Then, if p has some compact support, p has the smallest compact support.

PROOF. We firstly show that $\mathscr{M}(\mathscr{S})$ separates compact subsets of E in the following sense: *(sc)* if $K_0,K_1\subset E$ are disjoint nonvoid compact subsets, there is $f\in\mathscr{M}(\mathscr{S})$ such that $f=i$ on a neighborhood of K_i in E, for $i=0,1$. In fact, to begin with let us assume that $K_1=\{x_1\}$ is reduced to a point x_1. For every $t\in K_0$, choose $f_t\in\mathscr{M}(\mathscr{S})$ such that $f_t=0$ on an open neighborhood V_t of t in E, and $f_t=1$ on a neighborhood of x_1 in E. Select $t_1,\ldots,t_n\in K_0$ such that $K_0\subset V_{t_1}\cup\ldots\cup V_{t_n}$, and set $f=f_{t_1}\ldots f_{t_n}\in\mathscr{M}(\mathscr{S})$. We see that $f=0$ on a neighborhood of K_0 in E, and $f=1$ on a neighborhood of x_1 in E. Let finally K_1 be arbitrary. For every $t\in K_1$, choose $f_t\in\mathscr{M}(\mathscr{S})$ such that $f_t=0$ on a neighborhood of K_0 in E, and $f_t=1$ on an open neighborhood V_t of t in E. Select $t_1,\ldots,t_n\in K_1$ such that $K_1\subset V_{t_1}\cup\ldots\cup V_{t_n}$, and set $f=1-(1-f_{t_1})\ldots(1-f_{t_n})\in\in\mathscr{M}(\mathscr{S})$. We see that $f=i$ on a neighborhood of K_i, for $i=0,1$. This proves (*sc*).

We secondly show that, if $K_1,\ldots,K_n\subset E$ are compact supports of p, then $K=K_1\cap\ldots\cap K_n$ is a compact support of p. We may assume $n=2$. Let V be open in E containing K. Take $f\in\mathscr{M}(\mathscr{S})$ such that $f=0$ on an open subset V_1 of E containing K_1-V, and $f=1$ on an open subset V_2 of E containing K_2-V (by discarding the trivial case when K_1-V or K_2-V is empty). If then $g\in\mathscr{S}$ vanishes on V, write $g=fg+(1-f)g$, notice that $fg\in\mathscr{S}$ and fg vanishes on the neighborhood $V\cup V_1$ of K_1 in E, also that $(1-f)g\in\mathscr{S}$ and $(1-f)g$ vanishes on the neighborhood $V\cup V_2$ of K_2 in E, to deduce that $p(g)\leq p(fg)+p[(1-f)g]=0$, hence $p(g)=0$. This proves that K is a compact support of p.

Thirdly, let $\mathscr{K}$ be the nonvoid collection of compact supports of p, and K the compact intersection of $\mathscr{K}$. Let V be open in E containing K. By the finite intersection property, there are $K_1,\ldots,K_n\in\mathscr{K}$ such that $K_1\cap\ldots\cap K_n\subset V$. Hence, if $f\in\mathscr{S}$ vanishes on V, then $p(f)=0$, as $K_1\cap\ldots\cap K_n$ is a compact support of p. Thus K is also a compact support of p, clearly the smallest one. QED

REMARK 5. We may ask if, in Proposition 4, it is enough to assume that $\mathscr{M}(\mathscr{S})$ distinguishes points of E in the following sense: (*dp*) if $x_0,x_1\in E$, $x_0\neq x_1$, there is $f\in\mathscr{M}(\mathscr{S})$ such that $f=i$ at x_i, for $i=0,1$. The answer is negative. In fact, every p is supported by any nonvoid compact subset of E if and only if $\mathscr{S}$ has *uniqueness of continuation* in the sense that, if $f\in\mathscr{S}$ and $f^{-1}(0)$ has a nonvoid interior, then $f=0$. Sufficiency is clear. As to necessity, let $f_0\in\mathscr{S}$, $f_0\neq 0$, and $f_0^{-1}(0)$ have a nonvoid interior V. Choose a nonvoid compact subset K of V, for instance reduced to a point. Define p by $p(f)=\beta[f(a)]$ for $f\in\mathscr{S}$, where $a\in E$ is fixed so that $f_0(a)\neq 0$, and β is a continuous seminorm on F chosen so that $\beta[f_0(a)]>0$. It follows that p is not supported by K, proving necessity. Thus, if $\mathscr{S}=\mathscr{A}(U;\mathbf{K})\subset\mathscr{C}(U;\mathbf{K})$ is the algebra of all analytic $\mathbf{K}$-valued functions on the connected nonvoid open subset U of $\mathbf{R}^n$, or if $\mathscr{S}=\mathscr{H}(U;\mathbf{C})\subset\subset\mathscr{C}(U;\mathbf{C})$ is the algebra of all holomorphic $\mathbf{C}$-valued functions on the connected nonvoid open subset U of $\mathbf{C}^n$, then in both cases $\mathscr{S}$ satisfies (*dp*), but not (*sp*), and Proposition 4 breaks down for such $\mathscr{S}$, as a matter of fact.

LEMMA 6. Let $u:\mathscr{S}\to\mathscr{C}(E;G)$ be a linear mapping, where G is a real or complex Hausdorff locally convex space, and u is *local* in the sense that, if $f\in\mathscr{S}$, the interior of $f^{-1}(0)$ is contained in $u(f)^{-1}(0)$. If we endow $\mathscr{S}$ with the inverse image topology by u of the compact-open topology on $\mathscr{C}(E;G)$, then every continuous p has some compact support.

PROOF. Continuity of p means that there are a compact subset K of E and a continuous seminorm γ on G such that

$$p(f) \leq \sup_{x \in K} \gamma[u(f)(x)] \text{ for } f \in \mathscr{S}.$$

If $f \in \mathscr{S}$ vanishes on an open subset V of E containing K, then $u(f)$ vanishes on V too, hence on K, which implies $p(f)=0$. Thus K is a compact support of p. QED

COROLLARY 7. If $\mathscr{S}$ is endowed with the inverse image topology by a local linear mapping of a compact-open topology as in Lemma 6, and $\mathscr{M}(\mathscr{S})$ separates points of E as in Proposition 4, then every continuous p has the smallest compact support.

EXAMPLE 8. We know that there are some interesting variations of the continuous m-differentiability concept, such as those in the senses of Hadamard, or Fréchet, or other noteworthy senses, or even with additional nuclearity conditions (see [10] and other references in the Bibliography). The following example subsumes them all. Let E, F be real Hausdorff locally convex spaces not reduced to their origins, U be a nonvoid open subset of E, and $m=0,1,\ldots,\infty$. We denote by $\mathscr{C}^{(m)}(U;F)$ the vector space of all mappings $f: U \to F$ that are continuously m-differentiable in the following sense:

1) f is finitely m-differentiable, that is, for every vector subspace S of E of finite dimension, with S not reduced to its origin and $U \cap S$ nonvoid, we assume that the restriction $f|(U \cap S)$ is m-differentiable in the classical sense; hence we have the differential $d^k f: U \to \mathscr{L}_{as}({}^kE;F)$ with values in the vector space $\mathscr{L}_{as}({}^kE;F)$ of all symmetric k-linear mappings of E^k to F, for $k \in \mathbf{N}$, $k \leq m$.

2) $d^k f$ maps U into the vector space $\mathscr{L}_s({}^kE;F)$ of all continuous symmetric k-linear mappings of E^k to F, and $d^k f: U \to \mathscr{L}_s({}^kE;F)$ is continuous if $\mathscr{L}_s({}^kE;F)$ is endowed with the compact-open topology, for $k \in \mathbf{N}$, $k \leq m$.

We are going to deal with $\mathscr{S} = \mathscr{C}^{(m)}(U;F) \subset \mathscr{C}(U;F)$. Set

$$G = \prod_{k \in \mathbf{N},\, k \leq m} \mathscr{L}_s({}^kE;F)$$

endowed with the cartesian product topology, and consider the local linear mapping

$$u: f \in \mathscr{C}^{(m)}(U;F) \mapsto \left(\frac{1}{k!}\, d^k f\right)_{k \in \mathbf{N},\, k \leq m} \in \mathscr{C}(U;G)$$

to introduce on $\mathscr{C}^{(m)}(U;F)$ its compact-open topology of order m, which is the inverse image by u of the compact-open topology on $\mathscr{C}(U;G)$. We note that the algebra of multipliers of $\mathscr{C}^{(m)}(U;F)$ in $\mathscr{C}(U)$ contains each restriction $f(\phi)|U$, where $\phi \in E'$ is a continuous linear form on E, and $f \in \mathscr{C}^{(\infty)}(\mathbf{R};\mathbf{R})$. From Corollary 7 and the Hahn-Banach theorem, we conclude that every continuous seminorm on $\mathscr{C}^{(m)}(U;F)$ has a smallest compact support.

The preceding situation is more general than the Hadamard continuous m-differentiability. We may replace condition 2) in it by the following more stringent requirement:

2′) The mapping

$$(x, t) \in U \times E \mapsto d^k f(x) t^k \in F$$

is continuous, for $k \in \mathbf{N}$, $k \leq m$.

Sometimes we must deal with Fréchet continuous m-differentiability. This requires, among other things, enlarging the compact-open topology on $\mathscr{L}_s({}^kE; F)$ to the bounded-open topology for $k \in \mathbf{N}$, $k \leq m$ in the preceding conditions 1) and 2). We can even go further and use conditions 1) and 2), with the compact-open topology on $\mathscr{L}_s({}^kE; F)$ enlarged further to its projective-inductive topology, for $k \in \mathbf{N}$, $k \leq m$. Corollary 7 and the Hahn-Banach theorem still lead to the conclusion that every continuous seminorm on each of these new spaces $\mathscr{C}^{(m)}(U; F)$ has a smallest compact support.

Even more generally, motivated by examples in the Bibliography involving the use of nuclearity, as in [10] for instance, consider a vector subspace $\mathscr{S}$ of $\mathscr{C}^{(m)}(U; F)$ so that $\mathscr{M}(\mathscr{S})$ contains each restriction $f(\phi)|U$, where $\phi \in E'$ is a continuous linear form on E, and $f \in \mathscr{C}^{(\infty)}(\mathbf{R}; \mathbf{R})$. Let $\mathscr{D}_k$ be the image of $\mathscr{S}$ in $\mathscr{L}_s({}^kE; F)$ by $d^k: \mathscr{C}^{(m)}(U; F) \to \mathscr{L}_s({}^kE; F)$, and endow $\mathscr{D}_k$ with a natural Hausdorff locally convex topology, for $k \in \mathbf{N}$, $k \leq m$. Set

$$G = \prod_{k \in \mathbf{N},\, k \leq m} \mathscr{D}_k$$

endowed with the cartesian product topology, and consider the local linear mapping

$$u: f \in \mathscr{S} \mapsto \left(\frac{1}{k!}\, d^k f\right)_{k \in \mathbf{N},\, k \leq m} \in \mathscr{C}(U; G)$$

to introduce on $\mathscr{S}$ the inverse image topology by u of the compact-open topology on $\mathscr{C}(U; G)$. Thus, as before, from Corollary 7 and the Hahn-Banach theorem, we conclude that every continuous seminorm on $\mathscr{S}$ has the smallest compact support.

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